Toward a gauge theory for evolution equations on vector-valued spaces

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We investigate symmetry properties of vector-valued diffusion and Schrödinger equations. For a separable Hilbert space $H$ we characterize the subspaces of $L^2(\mathbb{R}^3;H)$ that are local, i.e., defined pointwise, and discuss the issue of their invariance under the time evolution of the differential equation. In this context, the possibility of a connection between our results and the theory of gauge symmetries in mathematical physics is explored. © 2009 American Institute of Physics. doi:10.1063/1.3227666

I. THE ABSTRACT SETTING: GLOBAL SYMMETRIES

In mathematical physics, one is often interested in the formulation of gauge theories. These are field theories where solutions of the relevant equations are symmetric, i.e., invariant under some transformation group of the functional values. A prototypical example is given by quantum electrodynamics, which is a gauge theory with respect to the symmetry group $U(1)$ (the unitary group) and leads to the introduction of the electromagnetic field.

The usual framework to deal with gauge theories in a mathematically rigorous way is that of differential geometry. The aim of this note is to propose a possible approach based on operator theoretic methods, instead, borrowing some ideas from the theory of vector bundles. We will only consider the Abelian case.

Let $H$ be a separable complex Hilbert space and consider the Bochner space $\mathcal{H} := L^2(\mathbb{R}^3;H)$, which is a Hilbert space with respect to the canonical inner product,

$$
(f|g) := \int_{\mathbb{R}^3} (f(x)|g(x))_H dx, \quad f, g \in \mathcal{H}.
$$

Let $\mathcal{V}$ be a Hilbert space which is densely embedded into $\mathcal{H}$—typically, a vector-valued Sobolev space. (Vector-valued Sobolev space can be defined in a standard way by means of scalar-valued test functions, see, e.g., Ref. 2, Sec. III.4.). Consider an $\mathcal{H}$-elliptic, continuous, sesquilinear form $a: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{C}$—see, e.g., Ref. 11, for the general theory of sesquilinear forms. Denote by $A$ the operator associated with $a$ and by $(e^{itA})_{t \geq 0}$ the semigroup generated by $A$. The form $a$ is symmetric if and only if $A$ is self-adjoint; in this case, by Stone’s theorem $iA$ generates a unitary group $(e^{itA})_{t \in \mathbb{R}}$. We assume throughout this note that $a$ is symmetric, i.e., $a(f,g) = \overline{a(g,f)}$ for all $f, g \in \mathcal{V}$. Thus, we will often simply refer to $a$ as to a quadratic form.

In mathematical language, a physical system with state space $\mathcal{H}$ is said to have a symmetry if there is a Lie group $\mathcal{O}$ of bounded linear operators on $\mathcal{H}$, such that each $O \in \mathcal{O}$ commutes with the time evolution of the system. Alternatively, in the Lagrangian formulation of classical field theory, a physical system with Lagrangian functional $\mathcal{L}$ admits a symmetry if there exists a Lie group $\mathcal{O} \subset \mathcal{L}(\mathcal{H})$, such that...

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for all \( O \in \mathcal{O} \) and all \( \phi \) smooth enough—say, in the domain of the sesquilinear form that appears in the weak formulation of the problem. It can be shown that these notions are equivalent in the self-adjoint case, cf. Proposition 1.4 or Ref. 5, Sec. 5, for a more detailed treatment.

Example 1.1: Let us sketch the prototypical case of the so-called Pauli equation, i.e., of the Schrödinger equation with magnetic vector potential \( M = (M_1, M_2, M_3) \) and electric scalar potential \( V \) for a particle in \( \mathbb{R}^3 \) with spin \( \frac{1}{2} \). In this context \( H = \mathbb{C}^2 \). Define the quadratic form

\[
\mathcal{L}(O\phi) = \mathcal{L}(\phi)
\]

(1.1)

for all \( O \in \mathcal{O} \) and all \( \phi \) smooth enough—in the domain of the sesquilinear form that appears in the weak formulation of the problem. It can be shown that these notions are equivalent in the self-adjoint case, cf. Proposition 1.4 or Ref. 5, Sec. 5, for a more detailed treatment.

Then the Lagrangian of the Pauli equation is given by

\[
\mathcal{L}(\phi) = \int_0^T i(\dot{\phi}(t)|\phi(t))_H + a(\phi(t))dt
\]

(1.3)

for all \( \phi \in C^1((0,T);L^2(\mathbb{R}^3;\mathbb{C}^2)) \cap C((0,T);H^1(\mathbb{R}^3;\mathbb{C}^2)) \). Observe that the time derivative commutes with any Lie group \( O \) of unitary operators acting on \( H = L^2(\mathbb{R}^3;\mathbb{C}^2) \), i.e.,

\[
i(\dot{\phi}(t)|\phi(t))_H = i(O\phi(t)|O\dot{\phi}(t))_H \quad \text{for all } O \in \mathcal{O}.
\]

Thus, we conclude that the system admits a symmetry in the sense of (1.1) if and only if

\[
O f \in \mathcal{V} \quad \text{and} \quad a(Of) = a(f) \quad \text{for all } f \in \mathcal{V} \quad \text{and} \quad O \in \mathcal{O}.
\]

(1.4)

Motivated by this example, we introduce the following.

Definition 1.2: A Lie group \( O \) of unitary operators on a Hilbert space \( H \) is said to be a global symmetry of the (possibly vector-valued) Schrödinger equation associated with the quadratic form \( a \) if condition (1.4) holds.

A special class of global symmetries has been introduced in Ref. 5: those unitary groups whose generator is \( i\mathcal{P} \), where \( \mathcal{P} \) is an orthogonal projection onto some closed subspace of \( H \), cf. also Ref. 4. We are particularly interested in a special class of orthogonal projections. To fix the ideas, let \( G \) be a closed subspace of \( H \) and introduce the closed subspace,

\[
\mathcal{G} := \{ f \in H; f(x) \in G \quad \text{for a.e. } x \in \mathbb{R}^3 \},
\]

(1.5)

of the Bochner space \( H = L^2(\mathbb{R}^3;H) \). We can consider the orthogonal projection \( \mathcal{P}_G \) of \( L^2(\mathbb{R}^3;H) \) onto \( \mathcal{G} \): one can easily check that this is given by

\[
(\mathcal{P}_G f)(x) = P_G f(x) \quad \text{for all } f \in L^2(\mathbb{R}^3;H) \quad \text{and a.e. } x \in \mathbb{R}^3,
\]

(1.6)

where \( P_G \) denotes the orthogonal projection of \( H \) onto \( G \). Being \( \mathcal{P}_G \) an orthogonal projection, \( i\mathcal{P}_G \) is skew adjoint, and by Stone’s theorem it generates a group \((e^{it\mathcal{P}_G})_{t \in \mathbb{R}}\) of unitary bounded linear
operators on $\mathcal{H}$. [In fact, since $P_G$ is bounded, it even generates an analytic group $(e^{tP_G})_{t \in \mathbb{R}}$.]

Remarks 1.3:

(1) It follows directly from the series expansion of the exponential function that
\[ e^{P} = e^{P} + (\text{Id} - P) = e^{P} + P^\perp, \quad z \in \mathbb{C}, \]  
(1.7)
for any orthogonal projection $P$ of a Hilbert space onto a closed subspace. (Here and in the following we denote by $P^\perp$ the orthogonal projection onto $\ker P$, i.e., $\text{Id} - P$.) Observe that for $z \in i\mathbb{R}$ it acts as $U(1)$ in the direction of $\text{range} P$ and as the identity in the orthogonal direction. Plugging (1.6) into (1.7) we obtain
\[ (e^{P}g)(x) = e^{P}(g)(x) + P^\perp(g)(x) = e^{P}(g)(x) + P^\perp(g)(x) = e^{P}g(f(x)) \]
for all $z \in \mathbb{C}$, $f \in L^2(\mathbb{R};H)$ and $x \in \mathbb{R}$. Thus, associated with $(e^{itP_G})_{t \in \mathbb{R}}$ we consider $(e^{itP_G})_{z \in \mathbb{R}}$. Clearly, also $(e^{itP_G})_{z \in \mathbb{R}}$ is unitary and, in fact, a (compact, simply connected) Lie group of dimension 1 with associated Lie algebra,
\[ \mathfrak{g} = \{iSP_G; s \in \mathbb{R}\}, \]
and $\Pi: e^{itP_G} \rightarrow e^{itP_G}$ is a unitary representation. Of course, there are as many Lie groups $(e^{itP_G})_{z \in \mathbb{R}}$ of the above type as closed subspaces $G$ of $H$. By the general theory of Lie groups, we know, in particular, that this representation is completely reducible, i.e., it is the direct sum of irreducible representations. In fact, by definition and due to (1.7), the representation $\Pi$ is irreducible if and only if the only closed subspaces $W$ of $L^2(\mathbb{R};H)$ satisfying $P_GW \subseteq W$ are the trivial ones. (This notion of irreducibility is clearly different from that of irreducibility in the sense of lattices, cf. Definition 2.4.) (Observe that for all closed subspaces $G \subseteq \mathcal{H}$ the three conditions $P_GW \subseteq W$, $P_WG \subseteq G$, and $P_GP_W = P_WP_G$ are equivalent, cf. Ref. 8, Lemma 2.3). In general, one sees that a necessary condition for irreducibility of the representation $\Pi$ is that the operator $P_G$ be irreducible in the sense of Banach lattices (in the case of $\dim H < \infty$, this amounts to saying that the matrix $P_G$ cannot be placed into block triangular form by permutations of rows and columns); irreducibility of $P_G$ is not sufficient though, since not all closed subspaces $G$ of $\mathcal{H}$ are of the form $G$ for a suitable subspace $G$ of $H$. Intuitively, the representation $\Pi: e^{itP} \rightarrow e^{itP}$ will not, in general, be irreducible, as one sees already in the simple case of $G = H = \mathbb{C}$ (i.e., $\Pi = \text{Id}$) if one considers the subspace $W$ of radial functions.

(2) One can also observe that by (1.7) each closed subspace $G$ defines canonically the circle bundle $S^1 \rightarrow S^1 \times H = S^1 \times G \times G^\perp \rightarrow H$. It is perhaps worth to remark that $U(1)$ is not the structure group (in the physical language, the gauge group) of this vector bundles—unless $G$ is trivial.

One can wonder whether, given a subspace $G$ of $H$, the subspace $G$ is invariant under $(e^{itP})_{t \in \mathbb{R}}$ provided that suitable conditions are verified by $A$, the (self-adjoint) operator associated with a quadratic form $a$. Such invariance properties can be characterized by a simple condition, as proved in Ref. 5, Sec. 5. Observe that the statement of Ref. 5, Proposition 5.3, refers to the case of an orthogonal projection whose range is a closed subspace defined as in (1.5). We repeat here its proof for the sake of self-containedness (and to fill a small gap in the proof in Ref. 5).

Proposition 1.4: Let $P$ be an orthogonal projection onto a closed subspace of a Hilbert space $\mathcal{H}$ and denote its orthogonal by $P^\perp$. Let $a: V \times V \rightarrow \mathbb{C}$ be an $\mathcal{H}$-elliptic, continuous, symmetric, densely defined sesquilinear form with associated operator $A$. Then the following assertions are equivalent.

(a) The ranges of $P$ and $P^\perp$ are invariant under $(e^{itP})_{t \in \mathbb{R}}$.
(b) If $\psi \in V$, then $P\psi \in V$ and $\text{Re} \ a(P\psi, \psi - P\psi) = 0$.
(c) The Lie groups $(e^{itP})_{t \in \mathbb{R}}$ and $(e^{itP^\perp})_{t \in \mathbb{R}}$ are global symmetries of the Schrödinger equation associated with the form $a$ in the sense of Definition 1.2.
Proof: Due to symmetry of the form, by Ouhabaz’s invariance criterion (cf. Ref. 8, Theorem 2.1, for a generalized version) (1.4) is equivalent to \( PV \subseteq V \) and \( a(\mathcal{P} \psi, (\text{Id} - \mathcal{P}) \psi) = 0 \) for every \( \psi \in V \). This is precisely (1.4).

By (1.7), \( V \) is invariant under \( \mathcal{P} \) if and only if it is invariant under the action of \( e^{it\mathcal{P}} \). Moreover, this formula implies

\[
a(e^{it\mathcal{P}} \psi, e^{it\mathcal{P}} \psi) = |e^{it} - 1|^2 a(\mathcal{P} \psi, \mathcal{P} \psi) + 2 \text{Re}(e^{it} - 1) a(\mathcal{P} \psi, \psi) + a(\psi, \psi).
\]

On the one hand, the identity \(|e^{it} - 1|^2 = 2 - 2 \text{Re} e^{it}\) shows that (1.4) implies if \( \psi \in V \), then \( e^{it\mathcal{P}} \psi \in V \) and \( a(\psi, \psi) = a(e^{it\mathcal{P}} \psi, e^{it\mathcal{P}} \psi) \) for all \( s \in \mathbb{R} \), i.e., (1.4) holds. On the other hand, if (1.4) holds, then the above calculation implies

\[
|e^{it} - 1|^2 a(\mathcal{P} \psi, \mathcal{P} \psi) = -2 \text{Re}(e^{it} - 1) a(\mathcal{P} \psi, \psi)
\]

for every \( s \in \mathbb{R} \). This is (1.4) for \( s = \pi \).

Let us revisit our motivating example and determine a class of global symmetries for the Pauli equation.

Example 1.5: Consider again the quadratic form introduced in (1.2). Fix a subspace \( G \) of \( \mathbb{C}^2 \) and consider the closed subspace \( H = L^2(\mathbb{R}^3; \mathbb{C}^2) \) defined as in (1.5). Denote by \( P_G \) and \( \mathcal{P}_G \) the orthogonal projections onto \( G \) and \( \mathcal{G} \), respectively. Clearly, if \( G = \mathbb{C}^2 \), then \( P_G \) is the identity and \( (e^{itP_G})_{t \in \mathbb{R}} = U(1) \) is trivially a global symmetry of the system. However, this is not the only one, as we will see.

In fact, we are going to prove that \( (e^{itP_G})_{t \in \mathbb{R}} \) and \( (e^{it\mathcal{P}_G})_{t \in \mathbb{R}} \) are global symmetries of the Pauli equation if and only if \( G, G^\perp \) are invariant under the matrices \( M_1(x), M_2(x), M_3(x), V(x) \) for a.e. \( x \in \mathbb{R}^3 \).

To begin with, observe that since \( P_G \) clearly leaves \( H^1(\mathbb{R}^3; \mathbb{C}^2) \) invariant, by Proposition 1.4, \( (e^{itP_G})_{t \in \mathbb{R}} \) and \( (e^{it\mathcal{P}_G})_{t \in \mathbb{R}} \) are global symmetries of the Pauli equation if and only if

\[
\text{Re} a(P_G \psi, \psi - \mathcal{P}_G \psi) = 0
\]

for all \( \psi \in V = H^1(\mathbb{R}^3; \mathbb{C}^2) \), i.e., if and only if

\[
\text{Re} \sum_{k=1}^3 \int_{\mathbb{R}^3} \left( \left( -i \frac{\partial}{\partial x_k} + M_k \right) P_G(f(x)) \left( -i \frac{\partial}{\partial x_k} + M_k \right) (f(x) - P_G(f(x))) \right) dx
\]

\[
+ \text{Re} \int_{\mathbb{R}^3} (V(x) (x) f(x) - P_G(f(x))) f(x) dx = 0
\]

for all \( f \in H^1(\mathbb{R}^3; \mathbb{C}^2) \) and a.e. \( x \in \mathbb{R}^3 \). Now, observe that \( P_G \) is space independent. Since the values of \( f(x), (\partial f/\partial x_1)(x), (\partial f/\partial x_2)(x), (\partial f/\partial x_3)(x) \) are mutually independent, by a localization argument this can be rephrased by saying that \( (e^{itP_G})_{t \in \mathbb{R}} \) is a global symmetry if and only if

\[
\text{Re} \sum_{k=1}^3 (y_k P_G^\perp y_k) + \text{Re} \sum_{k=1}^3 (-iP_G^\perp y_k M_k(x) P_G^\perp z_k) + \text{Re} \sum_{k=1}^3 (M_k(x) P_G^\perp z_k - iP_G^\perp y_k) + \text{Re} \sum_{k=1}^3 (y_k P_G M_k(x) P_G^\perp z_k)
\]

\[
+ \text{Re} \sum_{k=1}^3 (y_k P_G M_k(x) P_G^\perp z_k) + \text{Re} \sum_{k=1}^3 (M_k(x) P_G z_k - iP_G^\perp y_k) + \text{Re} (V(x) P_G z_k)
\]

vanishes for all \( y_1, y_2, y_3, z \in \mathbb{C}^2 \) and a.e. \( x \in \mathbb{R}^3 \). Thus, \( (e^{itP_G})_{t \in \mathbb{R}} \) is a global symmetry if and only if
\[
\text{Im} \sum_{k=1}^{3} (y_{1}(P_G M_k(x) P_G^0 + P_G^0 M_k(x) P_G) z)_{12} + \text{Re} \sum_{k=1}^{3} (M_k(x) P_G^0 | M_k(x) P_G^0) z)_{12} + \text{Re}(V(x) P_G^0 | P_G^0 z)_{12} = 0
\]

for all \(y_1, y_2, y_3, z \in \mathbb{C}^2\) and a.e. \(x \in \mathbb{R}^3\).

Due to linearity of the inner product, this condition holds if and only if both

\[
\sum_{k=1}^{3} (y_{1}(P_G M_k(x) P_G^0 + P_G^0 M_k(x) P_G) z)_{12} = 0
\]

and

\[
\text{Re} \sum_{k=1}^{3} (M_k(x) P_G^0 | P_G^0 z)_{12} + \text{Re}(V(x) P_G^0 | P_G^0 z)_{12} = 0
\]

independently of each other for all \(y_1, y_2, y_3, z \in \mathbb{C}^2\) and a.e. \(x \in \mathbb{R}^3\), i.e., if and only if

- \(P_G M_1(x) P_G^0 + P_G^0 M_1(x) P_G = 0\) for a.e. \(x \in \mathbb{R}^3\) and \(k=1,2,3\) and
- \(\text{Re} \sum_{k=1}^{3} (M_k(x) z_{12} | z_{12} z)_{12} + \text{Re}(V(x) z_{12} | z_{12} z)_{12} = 0\) for a.e. \(x \in \mathbb{R}^3\) and all \(z_1 \in G, z_2 \in G^\perp\).

Let us discuss the first condition. The first and second addend map \(H\) onto \(G\) and \(G^\perp\), respectively, hence we have obtained that \(P_G M_1(x) P_G^0 = 0\) and \(P_G^0 M_1(x) P_G = 0\), which is equivalent to the fact that \(M_1(x)\) leaves the spaces \(G, G^\perp\) invariant for a.e. \(x \in \mathbb{R}^3\) and \(k=1,2,3\).

If, however, \(G\) is left invariant under \(M_1(x)\) (and hence under \(M_1^2(x)\)) for a.e. \(x \in \mathbb{R}^3\) and \(k=1,2,3\), then the first addend in the second condition vanishes identically. Thus, we only have to observe that, for a.e. \(x \in \mathbb{R}^3\), \(\text{Re}(V(x) z_{12} | z_{12} z)_{12} = 0\) for all \(z_1 \in G, z_2 \in G^\perp\) if and only if \((V(x) z_{12} | z_{12} z)_{12} = 0\) for all \(z_1 \in G, z_2 \in G^\perp\), i.e., if and only if \(V(x)\) leaves invariant \(G\) (and \(G^\perp\) by self-adjointness) for a.e. \(x \in \mathbb{R}^3\).

If, for example, we consider the relevant case discussed in Example 1.1 with \(M_1=M_2=M_3 =0\) and \(V=\frac{1}{2} \beta \cdot \sigma\), the above results state that a one-dimensional subspace \(G=\{(1, \alpha)\} \subset \mathbb{C}^2\), \(\alpha \in \mathbb{C}\), defines a symmetry if and only if \(\beta \cdot \sigma((1, \alpha)) \subset \{(1, \alpha)\}\). A direct computation shows that this is equivalent to the condition

\[
\alpha(\beta_3 + \beta_1 - i \beta_2) = \beta_1 + i \beta_2 - \alpha \beta_3,
\]

i.e.,

\[
(\alpha^2 - 1) \beta_1 - i(\alpha^2 + 1) \beta_2 + 2 \alpha \beta_3 = 0.
\]

This is e.g., satisfied whenever \(\alpha = 1 \pm \sqrt{2}\) and \(\beta = (1, 0, -1)\).

Summarizing, we have proven that there exists a global symmetry if and only if the eigenspace decomposition of the matrix-valued potential \(V\) is space independent. Observe that this condition is slightly weaker than space independence of \(V\) itself, as it is can be seen considering any family of self-adjoint matrices,

\[
\begin{pmatrix}
1 & 0 \\
0 & f(x)
\end{pmatrix}, \quad x \in \mathbb{R}^3,
\]

for some \(f \in L^2(\mathbb{R}; \mathbb{C})\).

Remark 1.6: Let us shortly explain the formal connection of the results presented in Example 1.5 to the concept of symmetries, in particular, in the theory of Pauli equations for a nonrelativistic particle with spin \(\frac{1}{2}\) in \(\mathbb{R}^3\) as sketched in Example 1.1. In mathematical physics, symmetries are commonly discussed introducing a mapping

\[
\pi : \text{SO}(3) \to \text{SU}(2)
\]
and considering the transformation
\[ \psi(x) \mapsto \pi(R)\psi(R^{-1}x), \quad x \in \mathbb{R}^3, \]
cf. Ref. 6, Sec. IX.1.2, or Ref. 7, Problems 50 and 51. In the case of global symmetries, it is possible to separate the action of \( \pi(R) \) from the action of \( R^{-1} \).

In our formulation, we have just observed that each potential \( V \) of the form \( V = \frac{1}{2} \beta \cdot \sigma \) defines two symmetries on \( H = \mathbb{C}^2 \): a direct computation shows that they are given by the unitary groups generated by the orthogonal projections,
\[
P_\pm = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pm \frac{1}{2|\beta|} \begin{pmatrix} \beta_3 & \beta_1 - i\beta_2 \\ \beta_1 + i\beta_2 & -\beta_3 \end{pmatrix},
\]
ono onto the eigenspaces of \( V \). On the other hand, each symmetry on \( H = \mathbb{C}^2 \) is a one-parameter unitary group and can be formulated as \( (e^{i\lambda P})_{\lambda \in \mathbb{R}} \), where \( P \) is a self-adjoint operator. In this case we can assume the matrix \( P \) to take the form \( P = \frac{1}{2}(\text{Id} + \nu \cdot \sigma) \) for some \( \nu \in \mathbb{R}^3 \) with \(|\nu| = 1\).

The action of the symmetries on \( L^2(\mathbb{R}^3; \mathbb{C}^2) \) associated with the orthogonal projections introduced in (1.8) is given by
\[
\psi(x) \mapsto e^{i(s/2)x}e^{i(\nu/2)\sigma} \psi(x), \quad \psi \in L^2(\mathbb{R}^3; \mathbb{C}^2), x \in \mathbb{R}^3.
\]
Apart from the phase factor \( e^{i(s/2)x} \), this is nothing but a spin rotation of angle \( s \) about the axis given by the direction of the field \( \beta \).

II. SPACE-DEPENDENT SUBSPACES

The aim of this section is to describe and characterize the subspaces of the vector-valued function space \( \mathcal{H} = L^2(\mathbb{R}^3; H) \). In fact, we are interested in a more general setting than that presented in Sec. I. In contrast to the scalar-valued case, a new class of subspaces arises in the vector-valued context.

In this section we are going to study a kind of subspaces that are, roughly speaking, defined by an algebraic condition that has to be pointwise satisfied by functions’ values—but with this algebraic condition possibly changing from point to point. The following condition of localizability has been introduced in Ref. 13.

**Definition 2.1:** Let \( P_W \) be an orthogonal projection onto a closed subspace \( W \) of \( L^2(\mathbb{R}^3; H) \).

1. \( P_W \) is called localizable if
\[ \phi f \in W \quad \text{for all } f \in W \text{ and } \phi \in C^\infty_c(\mathbb{R}^3; \mathbb{C}). \]

2. \( P_W \) is called strictly local if there exists a family \( (P_x)_{x \in \mathbb{R}^3} \) of orthogonal projections onto closed subspaces of \( H \), such that \( x \mapsto P_x \) is strongly measurable and such that
\[
(P_W f)(x) = P_x(f(x)) \quad \text{for a.e. } x \in \mathbb{R}^3 \text{ and all } f \in L^2(\mathbb{R}^3; H). \tag{2.1}
\]

**Remark 2.2:** Since each orthogonal projection is a contraction, each strongly measurable orthogonal-projection-valued mapping \( \mathbb{R}^3 \ni x \mapsto P_x \in \mathcal{H} \) is essentially bounded and by Corollary 2.4, Ref. 3, defines in a canonical way an orthogonal projection onto a closed subspace of \( \mathcal{H} \). In other words, each strictly local orthogonal projection is localizable.

Does each orthogonal projection \( P_W \) onto some closed subspace \( W \) of \( L^2(\mathbb{R}^3; H) \) satisfy (2.1) for some family \( (P_x)_{x \in \mathbb{R}^3} \) of orthogonal projections onto closed subspaces of \( H \)? In general, the above question has a negative answer—simply think of the closed subspace of scalar-valued radial functions in \( L^2(\mathbb{R}^3; \mathbb{C}) \).

However, things change if we impose a mild locality assumption. The following result has been proven by Vogt in Ref. 13, Proposition 3. It essentially relies upon a characterization of local operators proved in Ref. 3, Sec. 2.

**Theorem 2.3:** Each localizable orthogonal projection is strictly local.
We address a problem similar to Vogt’s. Our aim is to characterize closed ideals of $\mathcal{H}$, i.e., those closed subspaces $I$ of $\mathcal{H}$ that satisfy the condition

$$f \in I \quad \text{and} \quad |g(x)|_{\mathcal{H}} \leq |f(x)|_{\mathcal{H}} \quad \text{for a.e.} \quad x \in \mathbb{R}^3 \quad \implies \quad g \in I.$$ 

(This condition only makes sense if $H$ is assumed to have a (complex) Hilbert lattice structure with absolute value $|\cdot|_{\mathcal{H}}$ what we do throughout—cf. Ref. 10, Sec. C-I. Recall that each Hilbert lattice is isometrically lattice isomorphic to an $L^2(\Omega; \mathcal{C})$-space for some measure space $\Omega$, see, e.g., Ref. 9, Corollary 2.7.5). We recall the following.

**Definition 2.4:** A bounded linear operator $A$ on a Hilbert lattice $\mathcal{H}$ is called irreducible in the sense of lattices if no nontrivial closed ideal of $\mathcal{H}$ is left invariant under $A$. A semigroup $(T(t))_{t \geq 0}$ of bounded linear operators is called irreducible in the sense of lattices if so is $T(t)$ for all $t \geq 0$.

**Remark 2.5:** In the scalar case ($H=\mathbb{C}$) it is well known that ideals of $L^2(\mathbb{R}^3; \mathbb{C})$ are exactly those spaces of the form $L^2(\omega; \mathbb{C})$ for $\omega \subset \mathbb{R}^3$, and irreducibility can be easily discussed by means of Ouhabaz’s invariance criterium. In particular, it is known that the heat semigroup on $L^2(\mathbb{R}^3; \mathbb{C})$ is irreducible; this follows from the general theory of Sobolev spaces, cf. Ref. 11, Theorem 4.5.

How can the characterization of closed ideals of scalar-valued function spaces be generalized to the vector-valued case? Clearly, performing such a generalization is necessary in order to discuss irreducibility of operators on $\mathcal{H}=L^2(\mathbb{R}^3; \mathcal{H})$.

Inspired by the notion of strict localizability, the main problem we address is whether for each closed ideal $I$ of $\mathcal{H}$ there exists a family of closed ideals $(I_x)_{x \in \mathbb{R}^3}$ of $H$, such that

$$I \doteq \{ f \in L^2(\mathbb{R}^3; \mathcal{H}); f(x) \in I_x \quad \text{for a.e.} \quad x \in \mathbb{R}^3 \}.$$ 

If we replace the word “ideal” by “subspace,” the answer to this problem is exactly Theorem 2.3. In fact, inspired by Vogt’s idea we obtain the following.

**Theorem 2.6:** Every orthogonal projection $P_x$ onto a closed ideal $I$ is strictly local. The ranges of the orthogonal projections $P_x$ appearing in (2.1) are, in fact, closed ideals of $\mathcal{H}$ for a.e. $x \in \mathbb{R}^3$.

Thus, in the following we may and do identify a projection $P$ onto a closed ideal of $\mathcal{H}$ and a family $(P_x)_{x \in \mathbb{R}^3}$ of projections onto closed ideals of $\mathcal{H}$.

**Proof:** Let us show that $P$ is localizable, i.e., that

$$\phi f \in I \quad \text{for all} \quad f \in I \quad \text{and} \quad \phi \in C^\omega_c(\mathbb{R}^3; \mathbb{C}).$$ 

To this end, it suffices to observe that

$$|\phi(x)f(x)| \leq \|\phi\|_\infty |f(x)| \quad \text{for a.e.} \quad x \in \mathbb{R}^3$$

and that $f$ and hence $\|\phi\|_\infty$ belong to $I$. It follows by the ideal property that $\phi f \in I$.

Then, we can apply Theorem 2.3 and deduce that (2.1) is satisfied by the orthogonal projection onto $I$ and by a family $(P_x)_{x \in \mathbb{R}^3}$ of orthogonal projections onto closed subspaces.

It remains to show that $P_x$ actually projects onto a closed ideal for a.e. $x \in \mathbb{R}^3$. To this end, recall the following general result: Given an orthogonal projection $Q$ of a Hilbert lattice $X$, its kernel is a closed ideal of $X$ if and only if $|Qy| = |Q||y|$ for all $y \in X$. [Recall that each Hilbert lattice is lattice isomorphic to some $L^2(\Omega; \mathcal{C})$-space, hence by Remark 2.5 its ideals are of the form $L^2(\Omega; \mathcal{C})$, $\mathcal{C} \subseteq \mathbb{C}$. Hence, the above mentioned result easily follows from the comments in Ref. 1, p. 94]. We are going to check this criterion for the kernel of a.e. $I-P_x$, i.e., for the range of a.e. $P_x$.

Let $f \in L^2(\mathbb{R}^3; \mathcal{H})$. Then

$$|f-P_xf(x)| = |(I-P)x|f(x) = (I-P)|f(x) = (I-P_x)|f(x)|$$

for a.e. $x \in \mathbb{R}^3$, i.e., $|(I-P)f| = (I-P)|f|$. Due to separability of $H$ and by suitable localization arguments, this suffices in order to show that $|(I-P)v| = (I-P)|v|$ for all $v \in H$. \qed
Conversely, the following holds.

**Proposition 2.7:** Consider a family \((P_x)_{x \in \mathbb{R}^3}\) of orthogonal projections onto closed ideals of \(H\) such that \(x \mapsto P_x\) is strongly measurable. Then the bounded linear operator \(\mathcal{P}\) defined via (2.1) is an orthogonal projection onto the closed ideal,

\[
\mathcal{I} = \{ f \in L^2(\mathbb{R}^3; H) : f(x) \in \text{range} P_x \text{ for a.e. } x \in \mathbb{R}^3 \}.
\]

**Proof:** It follows from Hölder’s inequality that \(\mathcal{P}\) is a bounded linear operator on \(L^2(\mathbb{R}^3; H)\). The facts that it is a projection and self-adjoint are consequences of the analogous properties of the operators \(P_x, x \in \mathbb{R}^3\).

\[\square\]

### III. LOCAL SYMMETRIES

With the aim of generalizing the notion of symmetry considered in Sec. I to the vector-valued case, it is natural to attach to each point \(x \in \mathbb{R}^3\) a closed subspace \(G_x\) of \(H\). We can thus define in analogy to the constant case a subspace,

\[
G = \{ f \in L^2(\mathbb{R}^3; H) : f(x) \in G_x \text{ for a.e. } x \in \mathbb{R}^3 \}, \tag{3.1}
\]

of \(H\). Can also such a space \(G\) be left invariant under the time evolution of the diffusion equation? This question can be answered by means of Proposition 1.4.

For the sake of simplicity we focus on the concrete case of the form \(a\) associated with the Laplace operator throughout this section.

More precisely, throughout the remainder of this section we consider an orthogonal-projection-valued mapping \(x \mapsto P_x := P_{G_x}\) to be of class \(H^1(\mathbb{R}^3, \mathcal{L}(H))\). Denote by \(\mathcal{P}_G\) the associated orthogonal projection of \(H\) onto \(G\).

**Remarks 3.1:**

1. **Observe that** \(P_x^\perp\) **is by assumption weakly differentiable and, in fact,**

\[
\nabla P_x^\perp = \nabla (\text{Id} - P_x) = -\nabla P_x \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{3.2}
\]

2. **Let us also mention the expression**

\[
\nabla P_x = d(P_x^\perp)(\nabla P_x)P_x + d(P_x)(\nabla P_x)P_x^\perp \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{3.3}
\]

**Formula (3.3) shows, in particular, that** \(\nabla P_x\) **(which, in general, is not a projection) boasts an off-diagonal block structure that is complementary to that of** \(P_x\). **This has been observed in Ref. 12, (1.15), in a different context.** [Here and in the following \(d(P_x), d(P_x^\perp)\) denote the diagonal \(3 \times 3\)-matrices \(P_x\) \(\text{Id}, \) i.e.,

\[
d(P_x) = \begin{pmatrix} P_x & 0 & 0 \\ 0 & P_x & 0 \\ 0 & 0 & P_x \end{pmatrix} \quad \text{and} \quad d(P_x^\perp) = \begin{pmatrix} P_x & 0 & 0 \\ 0 & P_x^\perp & 0 \\ 0 & 0 & P_x \end{pmatrix}.
\]

**Observe that this was not necessary in Ref. 12, (1.15), since the dependence of the operators \(P_x\) in Ref. 12, Sec. 1.1, is only on time, hence \(\nabla P_x\) is a scalar-valued function—rather than a vector field as in our context.]** It keeps its validity (with an analogous proof) in our framework, though.

3. **Combining (3.3) and** \(e^{zP_x} = e^{zP_x^\perp} + e^{zP_x}P_x^\perp\), **whose validity for all** \(z \in \mathbb{C}\) **and a.e.** \(x \in \mathbb{R}^3\) **can be proven as in (1.7), one obtains that**

\[
e^{z(P_x^\perp)(\nabla P_x)P_x} = e^{z(P_x^\perp)(\nabla P_x)P_x} + d(P_x^\perp)(\nabla P_x)P_x \tag{3.4}
\]

**for all** \(z \in \mathbb{C}\) **and a.e.** \(x \in \mathbb{R}^3\).
Recall the characterization of closed subspaces of $\mathcal{H} = L^2(\mathbb{R}^3; H)$ in Theorem 2.3.

**Proposition 3.2:** Let $\mathcal{P}_0 = (P_s)_{s \in \mathbb{R}}$ be the orthogonal projection onto a closed subspace $\mathcal{G}$ of $\mathcal{H}$. Consider the quadratic form $a$ associated with the Laplacian (without electromagnetic potential), i.e., a introduced in (1.2) with $M_1 = M_2 = M_3 = V = 0$. Then

$$a(e^{is\mathcal{P}_0}f) = \int_{\mathbb{R}^3} \| (e^{is} - 1)e^{-is\mathcal{P}_0}(\nabla P_s)f(x) + (\nabla f)(x) \|^2_{H^s} dx.$$ 

**Proof:** By (3.2)–(3.4) we can compute

$$\nabla (e^{is\mathcal{P}_0}f)(x) = (e^{is}P_s f(x) + P_s^+(f(x)) = e^{is}(\nabla P_s)f(x) + e^{is}d(P_s)(\nabla f)(x) + d(P_s^+)(\nabla f)(x)$$

$$= (e^{is} - 1)(\nabla P_s)f(x) + e^{is}d(P_s)(\nabla f)(x) + d(P_s^+)(\nabla f)(x)$$

$$= (e^{is} - 1)(\nabla P_s)f(x) + e^{is}d(P_s)(\nabla f)(x),$$

which holds for all $f \in \mathcal{V}$ and a.e. $x \in \mathbb{R}^3$. Accordingly, because $(e^{is\mathcal{P}_0})_{s \in \mathbb{R}}$ and hence $(e^{is\mathcal{P}_0})_{s \in \mathbb{R}}$ are unitary for a.e. $x \in \mathbb{R}^3$, the form $a$ associated with the Laplacian satisfies

$$a(e^{is\mathcal{P}_0}f) = \| (e^{is} - 1)e^{-is\mathcal{P}_0}f(x) + (\nabla f)(x) \|^2_{H^s} dx = a_s(f)$$

for all $s \in \mathbb{R}$ and all $f \in \mathcal{V}$. 

This shows that in the motivating Example 1.1 (but without electromagnetic potential), the Lagrangian $L(e^{is\mathcal{P}_0}\psi)$ stems from a Schrödinger equation with suitable potential that depends on $s$ in a $2\pi$-periodic fashion. In other words, we are led to considering a covariant derivative defined by

$$\nabla f := \nabla f + (e^{is} - 1)e^{-is\mathcal{P}_0}(\nabla P_s)f,$$

where the role of the gauge field is played by the multiplier

$$(e^{is} - 1)e^{-is\mathcal{P}_0}(\nabla P_s) = (e^{is\mathcal{P}_0} - e^{-is\mathcal{P}_0})(\nabla P_s).$$

**Remark 3.3:** Observe that the above computations bear some formal similarity to the theory of adiabatic perturbation theory, where a somewhat similar magnetic potential is expressed in terms of (time-dependent) spectral projections. Thus, it is tempting to interpret the functional

$$L_s(\phi) := \int_0^T \int_{\mathbb{R}^3} \| (e^{is} - 1)e^{-is\mathcal{P}_0}(\nabla P_s) \phi(t) \|^2_{H^s} dt$$

as an interaction Lagrangian and study the mutual asymptotic behavior of the time evolution of the systems associated with Lagrangians $L$ and $L + L_s$. However, two related problems soon arise: the quadratic form $a_s$ formally associated with the covariant derivative is not symmetric anymore, and accordingly the time evolution of the system associated with $L + L_s$ is not governed by a unitary group. In fact, the terms corresponding to the gauge field are $2\pi$-periodic off-diagonal perturbations of the leading term given by the free Hamiltonian $\Delta$ associated with $a = a_0$.

By Proposition 3.2 we are naturally led to introduce the following.

**Definition 3.4:** An orthogonal-projection-valued mapping $\mathbb{R}^3 \ni x \mapsto P_x \in \mathcal{L}(H)$ is called locally constant if there exists a family $(\omega_x)_{x \in \mathbb{R}^3}$ of open disjoint subsets of $\mathbb{R}^3$ such that $\mu(\mathbb{R}^3 \setminus \omega) = 0$, where $\omega := \bigcup_{x \in \mathbb{R}^3} \omega_x$ and $P_{\omega_0} = 0$, $P_{\omega_x} = \text{Id}$, and $P_{\omega_0}$ is the orthogonal projection onto a further closed ideal of $H$.

In the face of Proposition 1.4, we can complement Proposition 3.2 with the following observation.
Corollary 3.5: A necessary condition for the Lie group $(e^{itP})_{t \in \mathbb{R}}$ to be a global symmetry of the Schrödinger equation associated with the Laplacian is that the orthogonal-projection-valued mapping $\mathbb{R}^3 \ni x \mapsto P_x \in \mathcal{L}(H)$ be locally constant.

Proof: It follows by (3.4) that the gauge field can also be expressed as the multiplier,

$$(1 - e^{is})d(P)(\nabla P)P_\perp \oplus (e^{is} - 1)d(P_\perp)(\nabla P)P_\perp.$$

Such a gauge field vanishes identically, i.e., the covariant derivative satisfies $a_s(f) = a(f) = \|\nabla f\|^2$ for all $s \in \mathbb{R}$, if and only if the projections $P_\perp$ are $x$-independent (take, e.g., $s = \pi$).

Let us consider again the setting introduced in Proposition 3.2.

Corollary 3.6: Let $\Delta$ be the Laplace operator on $L^2(\mathbb{R}^3;H)$. The heat semigroup $(e^{t\Delta})_{t \geq 0}$ is irreducible (in the sense of lattices) if and only if $H = \mathbb{C}$.

Hence the Schrödinger equation associated with the Laplacian admits a symmetry defined by a projection onto a closed ideal of $\mathcal{H}$ if and only if $\dim H > 1$.

Observe that, in general, this is no more true as soon as the Laplacian is replaced by the full Schrödinger operator introduced in Example 1.1. In fact, nondiagonal potentials $M_1, M_2, M_3, V$ can provide a coupling effect that leads to irreducibility of the system.

Proof: If $H = \mathbb{C}$, the claimed characterization is well known, cf. Remark 2.5. If $H \neq \mathbb{C}$, take a one-dimensional closed ideal $I$ of $H$—say, the subspace spanned by the first vector of the Hilbert space basis of $H$. Then by Ouhazab's criterion the closed subspace $L^2(\mathbb{R}^3;I)$ of $\mathcal{H} = L^2(\mathbb{R}^3;H)$ is left invariant under $(e^{t\Delta})_{t \geq 0}$. The first claim follows.

The assertion concerning the Schrödinger equation is a consequence of Proposition 1.4.

Remarks 3.7:

(1) Recall that whenever an analytic semigroup $(T(t))_{t \geq 0}$ is positivity preserving (like, e.g., the heat semigroup generated by the Laplace operator) on some scalar-valued Lebesgue space, its irreducibility is equivalent to saying that positive initial data are instantaneously mapped into strictly positive solutions—another formulation of the linear heat equation’s well known infinite speed of propagation.

Observe that, even when the heat semigroup is not irreducible, for all $f \neq 0$ and all $t > 0$, the support

$$\text{supp } T(t)f := \{x \in \mathbb{R}^3 : T(t)f \neq 0 \text{ as an element of } H\}$$

is the whole space, just like in the scalar-valued case.

(2) The mapping $x \mapsto P_x$ naturally defines a vector bundle $\mathbb{R}^3 \to G \to H$, with $G$ as in (3.1). It is not clear to us whether the structure group associated with this vector bundle is related to the usual additive group on $\mathbb{R}^3$.

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